STABILITY OF SOLUTIONS OF A SYSTEM OF BOUNDARY LAYER EQUATIONS FOR A NONSTEADY FLOW OF INCOMPRESSIBLE FLUID

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The behavior of the solution of a system of equations for a nonsteady boundary layer is studied for a two-dimensional flow of incompressible fluid during an unlimited length of time. It is proved that under defined physical conditions, the longitudinal velocity component of the unsteady flow in the boundary layer tends to the longitudinal velocity component of a steady flow at $t \to \infty$.

We consider a system of boundary layer equations for a two-dimensional, nonsteady flow of a viscous incompressible fluid (see, for example [1])

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

in the region $D\ \{0 \leqslant t < \infty,\ 0 \leqslant x \leqslant x_0,\ 0 \leqslant y < \infty\}$ with the conditions

$$u|_{t=0} = u_0(x, y) \tag{2}$$

$$u|_{y=0} = 0, \quad v|_{y=0} = v_0(t,x), \quad u|_{x=0} = u_1(t, y), \quad \lim_{y\to\infty} u(t,x,y) = U(t,x)$$

where the functions $p(t, x)$ and $U(t, x)$ are related by Bernoulli's law

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} \tag{3}$$

We assume that when $t \to \infty$, the given functions $p(t, x)$, $U(t, x)$, and $v_0(t, x)$ tend uniformly, in terms of x, to the corresponding functions $p^\infty(x)$, $U^\infty(x)$, and $v_0^\infty(x)$, and $u_1(t, y)$ is, when $t > t_1$, where $t_1 \geqslant 0$ is some number, independent of t, i.e. when $t > t_1$, $u_1(t, y)$ coincides with some function $u_1^\infty(y)$. We consider a system of Prandtl equations for a steady boundary layer

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p^\infty}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4}$$

in the region $D^\infty\ \{0 \leqslant x \leqslant x_0,\ 0 \leqslant y < \infty\}$ with the conditions

$$u|_{y=0}=0, \quad v|_{y=0}=v_0^{\infty}(x), \quad u|_{x=0}=u_1^{\infty}(y), \quad \lim_{y\to\infty} u(x,y)=U^{\infty}(x) \quad (5)$$

We shall assume that the solution $u^{\infty}(x, y)$, and $v^{\infty}(x, y)$ of the system (4) with conditions (5) exists, and that it possesses the following properties: $\partial u^{\infty}/\partial y > 0$ when $0 \leqslant y < \infty$, and both $u^{\infty}(x, y)$ and $\partial u^{\infty}/\partial y$ have continuous and bounded derivatives, of the first order with respect to x and y, in D^{∞}; also, the derivatives $\partial^3 u^{\infty}/\partial y^3$ and $\partial v^{\infty}/\partial y$ exist.

We will also assume that the solution $u(t, x, y)$, and $v(t, x, y)$ of the system of equations (1) with conditions (2) and (3) exists, and has the following properties: $\partial u/\partial y > 0$ when $0 \leqslant y < \infty$, $u(t, x, y)$ and $\partial u/\partial y$ have continuous and bounded first order derivatives, with respect to t, x and y, in D; continuous derivatives $\partial^3 u/\partial y^3$, and $\partial v/\partial y$ exist and,

$$\left[\frac{\partial^3 u}{\partial y^3}\frac{\partial u}{\partial y} - \left(\frac{\partial^2 u}{\partial y^2}\right)^2\right]\left(\frac{\partial u}{\partial y}\right)^{-3} < K \tag{6}$$

in D, where K is some constant. These assumptions are always fulfilled even when physical limitations are imposed on the problems (1) to (3) as well as (4) and (5), provided x_0 is sufficiently small (see [2,and 3]).

In [2] a system of Prandtl equations (4) with the conditions (5), was studied. It was proved there that the solution of the problem (4) and (5) exists in the region D^{∞} for some $x_0 > 0$, provided that the functions $p^{\infty}(x)$, $v_0^{\infty}(x)$, $u_1^{\infty}(y)$, and $U^{\infty}(x)$ satisfy certain conditions of smoothness, that all agree at the point (0, 0) and that the conditions $u_1^{\infty}(y) > 0$ when $y > 0$ and $U^{\infty}(x) > 0$ when $x \geqslant 0$, are fulfilled. The solution $u^{\infty}(x, y)$, and $v^{\infty}(x, y)$ derived in the above-mentioned work is such, that $\partial u^{\infty}/\partial y > 0$ when $y \geqslant 0$, if $\partial u_1^{\infty}/\partial y > 0$ when $y \geqslant 0$.

In [3] a solution was derived of the system of boundary layer equations for the unsteady state (1) with conditions (2) and (3) in the region D for some $x_0 > 0$. Here some evenness of the functions $p(t, x)$, $U(t, x)$, $u_0(x, y)$, $v_0(t, x)$, and $u_1(t, y)$, is assumed, as well as their agreement with the equations of (1), and with the boundary conditions on the straight line $t = 0$, $y = 0$ and $t = 0$, $x = 0$. Also, it is assumed that $U(t, x) > 0$ and $u_0(x, y) > 0$ when $y > 0$, and $\partial u_1/\partial y > 0$ when $0 \leqslant y < \infty$.

This solution satisfies the condition

$$\frac{\partial u(t, x, y)}{\partial y} > 0 \quad \text{when } 0 \leqslant y < \infty$$

as well as the condition (6). We will show that

$$\lim_{t\to\infty} u(t, x, y) = u^{\infty}(x, y) \tag{7}$$

for all x, y in D^{∞}. This means that the solution $u(t, x, y)$ of the system of boundary layer equations for unsteady flow of a viscous incompressible fluid tends towards the solution $u^{\infty}(t, x)$ when $t \to \infty$, the latter solution corresponding to the problem of a system of boundary layer equations for the steady state. In particular, if follows from this, that when $t \to \infty$, the longitudinal velocity component $u(t, x, y)$ becomes stationary in the boundary layer during any perturbation of the steady solution, i.e. of the functions u^{∞}, v_0^{∞}, p^{∞}, u_1^{∞}, and U^{∞}, in the finite interval of time t.

In order to prove the relation (7) we eliminate v from the system (1) using the Crocco transform:

$$\tau = t, \quad \xi = x, \quad \eta = u(t, x, y)$$

Then the new unknown function $w = \partial u / \partial y$ satisfies the equation

$$\nu w^2 \frac{\partial^2 w}{\partial \eta^2} - \frac{\partial w}{\partial \tau} - \eta \frac{\partial w}{\partial \xi} + \frac{1}{\rho} \frac{\partial p}{\partial x} \frac{\partial w}{\partial \eta} = 0 \tag{8}$$

in the region $\Omega\, \{0 \leqslant \tau < \infty,\ 0 \leqslant \xi \leqslant x_0,\ 0 \leqslant \eta \leqslant U(\tau, \xi)\}$.

From the conditions (3) it follows that at the boundary of the region Ω the function w should satisfy the conditions

$$w|_{\tau=0} = \frac{\partial u_0}{\partial y} \equiv w_0(\xi, \eta), \quad w|_{\xi=0} = \frac{\partial u_1}{\partial y} \equiv w_1(\xi, \eta), \quad w|_{\eta=U(\tau,\xi)} = 0$$
$$\left(\nu w \frac{\partial w}{\partial \eta} - \frac{1}{\rho} \frac{\partial p}{\partial x} - v_0 w \right)\Big|_{\eta=0} = 0 \tag{9}$$

In Equations (4) and (5) we make an analogous transformation of the independent variables

$$\xi = x, \qquad \eta = u(x, y)$$

and introduce a new unknown function $w = \partial u / \partial y$.

The function w satisfies the equation

$$\nu w^2 \frac{\partial^2 w}{\partial \eta^2} - \eta \frac{\partial w}{\partial \xi} + \frac{1}{\rho} \frac{\partial p^\infty}{\partial x} \frac{\partial w}{\partial \eta} = 0 \tag{10}$$

in the region $\Omega^\infty \{0 \leqslant \xi \leqslant x_0,\ 0 \leqslant \eta \leqslant U^\infty(\xi)\}$, and the conditions

$$w|_{\xi=0} = \frac{\partial u_1^\infty}{\partial y} \equiv w_1^\infty(\eta), \qquad w|_{\eta=U^\infty(\xi)} = 0$$
$$\left(\nu w \frac{\partial w}{\partial \eta} - \frac{1}{\rho} \frac{\partial p^\infty}{\partial x} - v_0^\infty w \right)\Big|_{\eta=0} = 0 \tag{11}$$

on the boundary of the region Ω^∞. We will denote the solution of Equation (10) with the conditions (11), by $w^\infty(\xi, \eta)$.

We will consider the function $V(\tau, \xi, \eta) = w(\tau, \xi, \eta) - w^\infty(\xi, \eta)$, in the region Ω_1, which is the intersection of the region Ω with the cylinder $\{0 \leqslant \tau < \infty,\ 0 \leqslant \xi \leqslant x_0,\ 0 \leqslant \eta \leqslant U^\infty(\xi)\}$. Here w is the solution of the problem (8) and (9) and w^∞ is the solution of the problem (10) and (11). Subtracting Equation (10) for w^∞ from Equation (8), we obtain the equation for V

$$\nu (w^\infty)^2 \frac{\partial^2 V}{\partial \eta^2} - \frac{\partial V}{\partial \tau} - \eta \frac{\partial V}{\partial \xi} + \frac{1}{\rho} \frac{\partial p^\infty}{\partial x} \frac{\partial V}{\partial \eta} + \nu (w + w^\infty) \frac{\partial^2 w}{\partial \eta^2} V = \Phi(\tau, \xi, \eta)$$
$$\Phi(\tau, \xi, \eta) = \frac{1}{\rho} \left(\frac{\partial p^\infty}{\partial x} - \frac{\partial p}{\partial x} \right) \frac{\partial w}{\partial \eta} \tag{12}$$

The function $V(\tau, \xi, \eta)$ satisfies the conditions

$$V|_{\tau=0} = w_0(\xi, \eta) - w^\infty(\xi, \eta), \qquad V|_{\xi=0} = w_1(\tau, \eta) - w_1^\infty(\eta)$$
$$\left(\nu w^\infty \frac{\partial V}{\partial \eta} + \left(\nu \frac{\partial w}{\partial \eta} - v_0^\infty \right) V \right)\Big|_{\eta=0} = \Psi(\tau, \xi) \tag{13}$$
$$\Psi(\tau, \xi) = \left[\left(\frac{1}{\rho} \frac{\partial p}{\partial x} - \frac{1}{\rho} \frac{\partial p^\infty}{\partial x} \right) + (v_0 - v_0^\infty) w \right]\Big|_{\eta=0}$$

Since

$$\frac{\partial p(t,x)}{\partial x} \to \frac{\partial p^{\infty}(x)}{\partial x}, \qquad v_0(t,x) \to v^{\infty}{}_0(x) \text{ when } t \to \infty$$

uniformly with x, while w, $\partial w / \partial \eta$ are bounded in Ω, therefore $|\Phi(\tau, \xi, \eta)| < \varepsilon$ and $|\Psi(\tau, \xi)| < \varepsilon$, if $\tau > \tau_1$ and τ_1 is sufficiently great and ε is an arbitrary positive number. It is obvious that

$$V|_{\xi=0} = \frac{\partial u_1}{\partial y} - \frac{\partial u_1{}^{\infty}}{\partial y} = 0 \qquad \text{when } \tau > t_1$$

according to the previous assumption concerning $u_1(t, y)$.

Let $\varphi(s)$, which is twice continuously differentiable when $s \geqslant 0$, be equal to $3 - e^s$ when $0 \leqslant s \leqslant 1/2$ and be such that $1 \leqslant \varphi(s) \leqslant 3$ for all s.

We consider the function V_1, defined by the equality $V = V_1 e^{\beta\xi}\varphi(\alpha\eta)$, where $\alpha > 0$ and $\beta > 0$ are some sufficiently large numbers to the chosen below.

We shall show that $V_1(\tau, \xi, \eta) \to 0$ as $\tau \to \infty$ uniformly with respect to ξ and η and, consequently, $V(\tau, \xi, \eta) = w(\tau, \xi, \eta) - w^{\infty}(\xi, \eta) \to 0$ as $\tau \to \infty$.

From Equation (12) we obtain an equation for $V_1(\tau, \xi, \eta)$

$$L(V_1) \equiv \nu (w^{\infty})^2 \frac{\partial^2 V_1}{\partial \eta^2} - \frac{\partial V_1}{\partial \tau} - \eta \frac{\partial V_1}{\partial \xi} +$$

$$+ \left(\frac{1}{\rho} \frac{\partial p^{\infty}}{\partial x} + 2\nu\alpha (w^{\infty})^2 \frac{\varphi'}{\varphi} \right) \frac{\partial V_1}{\partial \eta} + c V_1 = \frac{\Phi e^{-\beta\xi}}{\varphi} \tag{14}$$

$$c \equiv \nu (w + w^{\infty}) \frac{\partial^2 w}{\partial \eta^2} - \eta\beta + \frac{\alpha}{\rho} \frac{\partial p^{\infty}}{\partial x} \frac{\varphi'}{\varphi} + \nu (w^{\infty})^2 \alpha^2 \frac{\varphi''}{\varphi}$$

If $\alpha\eta < 1/2$, then $-2 < \varphi' \leqslant -1$, $\varphi'' \leqslant -1$, and $1 \leqslant \varphi \leqslant 3$. By virtue of the properties of the solution $u^{\infty}(x, y)$ of the problem (4) and (5), the function $w^{\infty}(\xi, \eta) \geqslant a > 0$, where a is some constant, if $0 \leqslant \eta \leqslant \delta_1$, $\delta_1 > 0$ is sufficiently small and $\partial^2 w / \partial \eta^2$ is bounded. We shall choose $\alpha > 0$ large enough to satisfy

$$\nu (w + w^{\infty}) \frac{\partial^2 w}{\partial \eta^2} - \eta\beta + 2\frac{\alpha}{\rho} \left| \frac{\partial p^{\infty}}{\partial x} \right| - \frac{1}{3} \nu a^2 \alpha^2 < -M \qquad (M > 0)$$

Where the constant M can be chosen arbitrarily.

With α chosen in such a manner, the coefficient c in equation (14) is smaller than $-M$, if $\alpha\eta < 1/2$ and $\eta < \delta_1$.

We will further choose $\beta > 0$ so large that when $\eta > \min(1/2\,\alpha^{-1}, \delta_1)$ the coefficient c of V_1 in Equation (14) will be smaller than $-M$. The function V_1 satisfies the conditions

$$V_1|_{\tau=0} = (w_0(\xi, \eta) - w^{\infty}(\xi, \eta)) e^{-\beta\xi} \frac{1}{\varphi}, \qquad V_1|_{\xi=0} = (w_1(\tau, \eta) - w_1{}^{\infty}(\eta)) \frac{1}{\varphi} \tag{15}$$

$$l(V_1) \equiv \left(\nu w^{\infty} \frac{\partial V_1}{\partial \eta} - c_1 V_1 \right)\Big|_{\eta=0} = \frac{1}{2} \Psi e^{-\beta\xi}, \qquad c_1 \equiv \left(\frac{1}{2} \nu\alpha w^{\infty} - \nu \frac{\partial w}{\partial \eta} + v_0{}^{\infty} \right)\Big|_{\eta=0}$$

When choosing α, we may also assume

$$\alpha > \frac{2}{\nu a} \left(\max |v_0{}^{\infty}| + \nu \left| \frac{\partial w}{\partial \eta} \right| + 1 \right)$$

Hence, it can be assumed that, in the conditions (15), $c_1 > 1$.

If τ is sufficiently great, then $|U(\tau, \xi) - U^\infty(\xi)| \leqslant \varkappa$, where $\varkappa > 0$ is an arbitrary known number. Therefore

$$|V| = |w - w^\infty| \leqslant \varepsilon \text{ when } \eta > U^\infty(\xi) - \varkappa, \quad \tau > \tau_2$$

provided $\varkappa$ and τ_2^{-1} are sufficiently small, since the function $w^\infty(\xi, \eta) \to 0$ as $U^\infty(\xi) - \eta \to 0$ and $w(\tau, \xi, \eta) \to 0$ as $U(\tau, \xi) - \eta \to 0$ uniformly with τ. Here, $\varepsilon > 0$ is any given number. Consequently

$$|V_1| = \left|Ve^{-\beta\xi}\frac{1}{\varphi}\right| \leqslant \varepsilon \quad \text{when } \tau > \tau_2, \ \ \eta > U^\infty(\xi) - \varkappa$$

We shall denote the part of the region Ω_1, for which $\tau \geqslant \sigma$, by G_σ. Let $\delta > 0$ be an arbitrary given number. We will show that, in Ω_1,

$$|V_1(\tau, \xi, \eta)| \leqslant \delta + M_1 e^{-\gamma\tau} \tag{16}$$

where $\gamma > 0$ is any constant less than M; the constant $M_1 > 0$ depends upon δ and γ. We consider in G_σ the functions $W_\pm$:

$$W_+ = \delta + M_1 e^{-\gamma\tau} + V_1, \quad W_- = \delta + M_1 e^{\gamma\tau} - V_1$$

Here δ and γ are already given, while M_1 shall be chosen below. The functions W_+, and W_- satisfy the equation

$$L(W_\pm) = cM_1 e^{-\gamma\tau} + \gamma M_1 e^{-\gamma\tau} + c\delta \pm \Phi e^{-\beta\xi}\frac{1}{\varphi} \tag{17}$$

We note that $c < -M$ and $\gamma < M$. Therefore the sum of the first two terms in the right-hand side of Equation (17) is negative. Since $|\varphi^{-1}\Phi e^{-\beta\xi}| \to 0$ as $\tau \to \infty$ uniformly with ξ and η, then

$$\pm \Phi e^{-\beta\xi}\varphi^{-1} + c\delta < 0 \text{ when } \delta > 0$$

if τ is sufficiently great. This shows that $L(W_+) < 0$ and $L(W_-) < 0$ in G_σ, if σ is sufficiently great.

From the inequality $L(W_\pm) < 0$ in G_σ it follows that $W_\pm$ cannot have a negative minimum inside the region G_σ or when $\xi = x_0$, and also on the secant $\tau = \tau_3$, $\tau_3 > \sigma$, if $W_\pm$ is considered for $\sigma < \tau < \tau_3$.

We shall also show that $W_\pm$ cannot have a negative minimum on the remaining part of the boundary of G_σ, if σ is sufficiently great, i.e. $W_+ \geqslant 0$ and $W_- \geqslant 0$ in G_σ. When $\xi = 0$, we have $W_\pm \geqslant 0$ also on this part of the boundary region of G_σ, which lies on the surface $\eta = U^\infty(\xi)$ or $\eta = U(\tau, \xi)$, since $V_1|_{\xi=0} = 0$ for sufficiently large τ and $|V_1| < \varepsilon$ when $\eta > U(\xi) - \varkappa$ and $\tau > \tau_2$. Let us choose $\varepsilon < \delta$ and sufficiently small corresponding $\varkappa$ and τ_2^{-1}.

When $\eta = 0$, we have

$$l(W_\pm) = -c_1(\delta + M_1 e^{-\gamma\tau}) \pm {}^1/_2\Psi e^{-\beta\xi} < 0$$

provided $\tau > \tau_4$ and τ_4 is sufficiently great, since $c_1 > 1$, and $|\Psi e^{-\beta\xi}| \to 0$ as $\tau \to \infty$ uniformly with respect to ξ. Thus $W_\pm$ cannot have a negative minimum when $\eta = 0$ and $\tau > \tau_4$. We will choose M_1 large enough to fulfil the inequalities $W_+ > 0$ and $W > 0$, when

$\tau = \sigma\ (\sigma > \tau_2, \sigma > \tau_4)$.

Thus, everywhere in G_σ, provided σ is sufficiently large, $W_\pm \geqslant 0$ and, consequently, $\pm V_1 + M_1 e^{-\gamma\tau} + \delta \geqslant 0$,

$$|V_1| \leqslant \delta + M_1 e^{-\gamma\tau} \quad \text{in } G_\sigma \tag{18}$$

It is obvious that having increased, if necessary, the constant M_1, we obtain from (18), that the inequality (16) is true in Ω_1 for some constant M_1. From (18) it follows that $w(\tau, \xi, \eta) \to w^\infty(\xi, \eta)$ uniformly with respect to ξ and η when $\tau \to \infty$. Hence, it follows that

$$\lim_{t\to\infty} u(t, x, y) = u^\infty(x, y) \tag{19}$$

for all x and y in D^∞.

Indeed, $|U^\infty(x) - u^\infty(x, y)| < \varepsilon$ and $|U(t, x) - u(t, x, y)| < \varepsilon$ for $y > y_1$, since $w(\tau, \xi, \eta) \to 0$ when $U(\tau, \xi) - \eta \to 0$ uniformly with respect to τ. Thus

$$|u^\infty(x, y) - u(t, x, y)| < 2\varepsilon \tag{20}$$

for sufficiently large t and $y > y_1$.

When $y \leqslant y_1 < \infty$ the inequality (20) results, for large t, from the representation of $u^\infty(x, y)$ and $u(t, x, y)$ by functions of w^∞ and w. Indeed

$$y = \int_0^{u(t,x,y)} \frac{ds}{w(t, x, s)}, \qquad y = \int_0^{u^\infty(x,y)} \frac{ds}{w^\infty(x, s)}$$

We have

$$0 = \int_0^{u(t,x,y)} \frac{ds}{w(t, x, s)} - \int_0^{u^\infty(x,y)} \frac{ds}{w^\infty(x, s)} = \int_0^{u^\infty(x,y)} \left(\frac{1}{w} - \frac{1}{w^\infty}\right) ds + \int_{u^\infty(x,y)}^{u(t,x,y)} \frac{ds}{w(t, x, s)}$$

Since

$$U(t, x) - u(t, x, y) > \varkappa_1, \quad U^\infty(x) - u^\infty(x, y) > \varkappa_1 \quad \text{for } y \leqslant y_1$$

and, besides

$$w(t, x, s) \geqslant a_1 > 0, \quad w^\infty(x, s) \geqslant a_1 > 0 \quad \text{for } s < U(t, x) - \varkappa_1,\ s < U^\infty(x) - \varkappa_1$$

hence, we have

$$u(t, x, y) - u^\infty(x, y) = w(t, x, s_1) \int_0^{u^\infty(x,y)} \frac{(w - w^\infty)}{w w^\infty} ds \qquad \text{for } y \leqslant y_1$$

Where s_1 is confined between $u(t, x, y)$ and $u^\infty(x, y)$. Hence, it follows that

$$|u(t, x, y) - u^\infty(x, y)| \leqslant \delta_2 + M_2 e^{-\gamma t} \quad \text{for} \quad y \leqslant y_1$$

and for some positive constants δ_2 and M_2; δ_2 can be taken to be arbitrarily small, M_2 varies with δ_2 and y_1. Thus $u(t, x, y) \to u^\infty(x, y)$ as $t \to \infty$ for $0 \leqslant x \leqslant x_0$, $0 \leqslant y < \infty$.

Note 1. If the functions $p(t, x)$, $U(t, x)$, $v_0(t, x)$, and $u_1(t, y)$ are such that when $t > t_1$ they do not depend on t, then we easily see that it is possible to prove the inequality (16) when $\delta = 0$. In such a case we have

$$|w(\tau, \xi, \eta) - w^{\infty}(\xi, \eta)| \leqslant M_1 e^{-\gamma t} \tag{21}$$

in Ω, where γ is any number and M_1 is some constant, depending upon γ. From the inequality (21) it follows that, in this case,

$$|u(t, x, y) - u^{\infty}(x, y)| \leqslant M_2 e^{-\gamma t}, \quad \text{if} \quad y \leqslant y_1 < \infty, \quad 0 \leqslant x \leqslant x_0$$

Here, M_2 depends upon γ and y_1.

Note 2. Above it was assumed, that $u_1(t, y)$ in the conditions (3) are independent of t for sufficiently large t. The equality (16) will also be true if this assumption is replaced with the condition that

$$\left| \frac{\partial u_1}{\partial y} - \frac{\partial u_1^{\infty}}{\partial y} \right| \to 0 \qquad \text{as} \quad t \to \infty$$

uniformly with respect to y and $|u_1^{-1} - u_1^{\infty-1}| \to 0$ as $t \to \infty$ uniformly with respect to η, where u_1^{-1}, $u_1^{\infty-1}$ are the corresponding inverse functions for u_1 and u_1^{∞}, i.e.

$$\eta = u_1(t, y), \ y = u_1^{-1}(t, \eta) \text{ and } \eta = u_1^{\infty}(y), \ y = u_1^{\infty-1}(\eta)$$

In this case

$$|V|_{\xi=0} = \left| \frac{\partial u_1(t, y)}{\partial y} - \frac{\partial u_1^{\infty}(y)}{\partial y} \right| = \left| \frac{\partial u_1(t, u_1^{-1}(t, \eta))}{\partial y} - \frac{\partial u_1^{\infty}(u_1^{\infty-1}(\eta))}{\partial y} \right| \leqslant$$

$$\leqslant \left| \frac{\partial u_1(t, u_1^{-1}(t, \eta))}{\partial y} - \frac{\partial u_1^{\infty}(u_1^{-1}(t, \eta))}{\partial y} \right| + \max \left| \frac{\partial^2 u_1^{\infty}}{\partial y^2} \right| |u_1^{-1}(t, \eta) - u_1^{\infty-1}(\eta)|$$

Hence, from the above assumptions, it follows that $|V|_{\xi=0} \to 0$ as $t \to \infty$ uniformly with respect to η, since $\partial^2 u_1^{\infty} / \partial y^2$ is bounded when $0 \leqslant y < \infty$. This is sufficient for the proof of the inequalities (16) and, consequently, (19).

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